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# Bases in Banach spaces of smooth functions on Cantor-type sets

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**Abstract**

We suggest a Schauder basis in Banach spaces of smooth functions and traces of smooth functions on Cantor-type sets. In the construction, local Taylor expansions of functions are used.


*Keywords:* Topological bases; $C^p$-spaces; Cantor sets; Taylor expansions

## 1. Introduction

We consider the basis problem for Banach spaces of differentiable functions. It is not difficult to present a (Schauder) basis in the space $C^p[0, 1]$. Indeed, by means of the operator $T : C[0, 1] \longrightarrow C^p_{\mathcal{F}}[0, 1] : f \mapsto \int_0^x \int_0^{x_1} \cdots \int_0^{x_{p-1}} f(x_p)\mathrm{d}x_p \cdots \mathrm{d}x_1$ we have an isomorphism $C^p[0, 1] \simeq \mathbb{R}^p \oplus C[0, 1]$. Here $C^p_{\mathcal{F}}[0, 1]$ denotes the subspace of functions that are flat at 0, that is such that $g^{(k)}(0) = 0$ for $0 \le k \le p - 1$. Therefore, any Schauder basis in $C[0, 1]$ gives a corresponding basis in the space $C^p[0, 1]$.

For other compact sets $K$, the question about a basis in the space $C^p(K)$ may be much more difficult. For example, one of the basis problems of Banach concerning the space $C^1[0, 1]^2$ (see [1, p.147]) was solved only 37 years later by Ciesielski in [3] and Schonefeld in [14]. Even after this, a generalization to the case $C^p[0, 1]^2$ with $p \ge 2$ was not trivial (see [15] for details).

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Schauder bases in the spaces $C^p[0,1]^q$ were suggested independently by Ciesielski and Domsta in [4] and by Schonefeld in [15]. We should notice that two main approaches in the construction of bases were presented in these papers. Schonefeld's system is interpolating basis, while the basis constructed in [4] is orthonormal, but not interpolating.

Mitjagin established in [13, Th.3] that if $M_1$ and $M_2$ are $n$-dimensional smooth manifolds with or without boundary, then the spaces $C^p(M_1)$ and $C^p(M_2)$ are isomorphic. This result essentially enlarges the class of compact sets $K$ with a basis in the space $C^p(K)$, but it cannot be applied to compact sets with infinitely many components, in particular for nontrivial totally disconnected sets.

Jonsson considered in [9] triangulations of compact sets in $\mathbb{R}$ and constructed an interpolating Schauder basis in the space $C^p(K)$ provided the compact set $K$ admits a sequence of regular triangulations. By Theorem 1 in [9], the last condition is valid if and only if $K$ preserves the so-called Local Markov Inequality, which in turn means that $K$ is uniformly perfect [11, Section 2.2]. On the other hand, the space considered in [9] was actually $\mathcal{E}^p(K)$, that is the Whitney space of functions on $K$ extendable to functions from $C^p(\mathbb{R})$, but equipped with the norm of the space $C^p(K)$. It should be noted that, in general, the space $\mathcal{E}^p(K)$ is not complete in this norm (see [9, p.54] and Section 3).

Here we consider the case of a Cantor-type set $K$ and present explicitly a Schauder basis in the Banach space $C^p(K)$ of $p$ times differentiable on $K$ functions as well as in the Whitney space $\mathcal{E}^p(K)$. In the construction local Taylor expansions of functions are used. In a sense, this generalizes the basis from Haar functions in the space $C(K)$ for the Cantor set $K$ [16, Prop. 2.2.5]. Clearly, the system of monomials cannot form a basis in the space $C^p[0,1]$ with $p \le \infty$, containing non-analytic functions. In our case, for a Cantor-type set $K$, "local Taylor" bases are presented only in the Banach spaces $\mathcal{E}^p(K)$ with $p < \infty$, but not in the Fréchet spaces $\mathcal{E}(K)$ of Whitney functions of infinite order. For the last case, a basis was suggested in [6] by means of local Newton interpolations; see also [7] for a similar basis in $C(K)$. Interpolating Schauder bases in other functional Banach spaces on fractals were given in [10]. It should be noted that not all functional spaces possess interpolating bases [8].

## 2. Local Taylor expansions on Cantor-type sets

Given compact set $K \subset \mathbb{R}$, $f = (f^{(k)})_{0\le k\le n} \in \prod_{0\le k\le n} C(K)$ and $a, x \in K$, let us consider the formal Taylor polynomial $T_a^n f(x) = \sum_{0\le k\le n} f^{(k)}(a)\frac{(x-a)^k}{k!}$ and the corresponding Taylor remainder $R_a^n f(x) = f(x) - T_a^n f(x)$. In the case of perfect $K$, the set $(f^{(k)}(x))_{0\le k\le n, x\in K}$ is completely defined by the values of $f$ on $K$ provided existence of the corresponding derivatives. If $m \le n$ and $a, b, c \in K$ then trivially

$$T_a^n \circ T_b^m = T_b^m, \qquad R_a^n \circ R_b^m = R_a^n, \qquad R_a^n \circ T_b^m = 0. \tag{1}$$

Let $\Lambda = (l_s)_{s=0}^{\infty}$ be a sequence such that $l_0 = 1$ and $0 < 2l_{s+1} < l_s$ for $s \in \mathbb{N}_0 := \{0, 1, \ldots\}$. Let $K(\Lambda)$ be the Cantor set associated with the sequence $\Lambda$ that is $K(\Lambda) = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0,1]$, $E_s$ is a union of $2^s$ closed *basic* intervals $I_{j,s} = [a_{j,s}, b_{j,s}]$ of length $l_s$ and $E_{s+1}$ is obtained by deleting the open concentric subinterval of length $h_s := l_s - 2l_{s+1}$ from each $I_{j,s}$, $j = 1, 2, \ldots 2^s$.

Let us consider the set of all left endpoints of basic intervals. Since $a_{j,s} = a_{2j-1,s+1}$ for $j \le 2^s$, any such point has infinitely many representations in the form $a_{j,s}$. We select the representation with the minimal second subscript and call it the *minimal representation*. If $j$

is even, then the representation $a_{j,s}$ is minimal for the corresponding point. Otherwise, for $j = 2^q(2m+1)+1 > 1$ we obtain $a_{j,s} = a_{2m+2,s-q}$. Clearly, $a_{1,s} = a_{1,0}$ for all $s$. Therefore we have a bijection between the set of all left endpoints of basic intervals and the set $A = a_{1,0} \cup (a_{2j,s})_{j=1,s=1}^{2^{s-1},\infty}$.

Let us enumerate the set $A$ by first increasing $s$, then $j$: $x_1 = a_{1,0} = 0$, $x_2 = a_{2,1} = 1-l_1$, $x_3 = a_{2,2} = l_1 - l_2$, $x_4 = a_{4,2} = 1-l_2, \ldots$ and, in general, $x_{2^s+k} = a_{2k,s+1}$ for $k = 1, 2, \ldots, 2^s$.

Let us fix $p \in \mathbb{N}$. For $s \in \mathbb{N}_0$, $j \le 2^s$ and $0 \le k \le p$ let $e_{k,j,s}(x) = (x - a_{j,s})^k/k!$ if $x \in K(\Lambda) \cap I_{j,s}$ and $e_{k,j,s} = 0$ on $K(\Lambda)$ otherwise. Given $f = (f^{(k)})_{0\le k\le p} \in \prod_{0\le k\le p} C(K(\Lambda))$, let $\xi_{k,j,s}(f) = f^{(k)}(a_{j,s})$ for the same values of $s$, $j$, and $k$ as above. Clearly, for the fixed level $s$, the system $(e_{k,j,s}, \xi_{k,j,s})$ is biorthogonal, that is $\xi_{k,j,s}(e_{n,i,s}) = \delta_{kn} \cdot \delta_{ij}$. In order to obtain biorthogonality as well with regard to $s$, we will use the following convolution property of the values of functionals on the basis elements (see [5, L.3.1] and [6, L.2]). Let $I_{i,n} \supset I_{j,s-1}$. Then

$$\sum_{m=k}^{p} \xi_{k,2j,s}(e_{m,j,s-1}) \cdot \xi_{m,j,s-1}(e_{q,i,n}) = \xi_{k,2j,s}(e_{q,i,n}) \quad \text{for all } q \le p.$$

Indeed, $(e_{k,i,n})_{k=0}^p$, $(e_{k,j,s-1})_{k=0}^p$, $(e_{k,2j,s})_{k=0}^p$ are three bases in the space $\mathcal{P}_p(I_{2j,s})$ of polynomials of degree not greater than $p$ on the interval $I_{2j,s}$. If $M_{r\leftarrow t}$ denotes the transition matrix from the $t$-th basis to the $r$-th basis, then the identity above means $M_{3\leftarrow 2}M_{2\leftarrow 1} = M_{3\leftarrow 1}$.

On the other hand, in our case, this identity is the corresponding binomial expansion:

$$\sum_{m=k}^{q} \frac{(a_{2j,s} - a_{j,s-1})^{m-k}}{(m-k)!} \cdot \frac{(a_{j,s-1} - a_{i,n})^{q-m}}{(q-m)!} = \frac{(a_{2j,s} - a_{i,n})^{q-k}}{(q-k)!}.$$

Here we consider summation until $q$ since for $q < m \le p$, the terms $\xi_{m,j,s-1}(e_{q,i,n})$ vanish.

We restrict our attention only to the functions $(e_{k,1,0})_{k=0}^p$ and $(e_{k,2j,s})_{k=0,j=1,s=1}^{p,2^{s-1},\infty}$ corresponding to the set $A$. Let us enumerate this family in the lexicographical order with respect to the triple $(s, j, k)$ : $f_n = e_{n-1,1,0} = \frac{1}{(n-1)!}(x - x_1)^{n-1} \cdot \chi_{1,0}$ for $n = 1, 2, \ldots, p+1$. Here and in what follows, $\chi_{j,s}$ denotes the characteristic function of the interval $I_{j,s}$. After this, $f_n = e_{n-p-2,2,1} = \frac{1}{(n-p-2)!}(x - x_2)^{n-p-2} \cdot \chi_{2,1}$ for $n = p+2, p+3, \ldots, 2(p+1)$ and in general, if $(m-1)(p+1)+1 \le n \le m(p+1)$, then $f_n = \frac{1}{k!}(x - x_m)^k \cdot \chi_{2i,s+1} = e_{k,2i,s+1}$. Here $m = 2^s + i$ with $1 \le i \le 2^s$ and $k = n - (m-1)(p+1) - 1$. We see that all functions of the type $\frac{1}{k!}(x - x_m)^k \cdot \chi_{2i,s+1}$ with $0 \le k \le p$ and $m = 2^s + i \in \mathbb{N}$ are included into the sequence $(f_n)_{n=1}^\infty$.

For the same values of parameters as above, we define the functionals $\eta_{k,1,0} = \xi_{k,1,0}$ for $k = 0, 1, \ldots, p$ and

$$\eta_{k,2j,s} = \xi_{k,2j,s} - \sum_{m=k}^{p} \xi_{k,2j,s}(e_{m,j,s-1}) \cdot \xi_{m,j,s-1}$$

for $s \in \mathbb{N}$, $j = 1, 2, \ldots, 2^{s-1}$, and $k = 0, 1, \ldots, p$. In what follows, we will use the minimal representations of the points $a_{j,s}$ and the corresponding functionals $\xi_{m,j,s}$. For example, $\eta_{k,2,s} = \xi_{k,2,s} - \sum_{m=k}^{p} \xi_{k,2,s}(e_{m,1,0}) \cdot \xi_{m,1,0}$. This agreement is justified by the fact that the value $\xi_{m,j,s}(f) = f^{(m)}(a_{j,s})$ does not depend on the representation of the point $a_{j,s}$ and the functions $e_{m,j,s-1}, e_{m,r,s-q}$ coincide on the interval $I_{2j,s}$ if $a_{j,s-1} = a_{r,s-q}$.

The crucial point of the construction is that the functionals $\eta_{k,2j,s}$ are biorthogonal, not only to all elements $(e_{k,2j,s-1})_{k=0}^{p}$, but also, by the convolution property, to all $(e_{k,2i,n})_{k=0}^{p}$ with $n = 0, 1, \ldots, s-2$ and $i = 1, 2, \ldots, 2^{n-1}$. In addition, the functional $\eta_{k,2j,s}$ takes zero value at all elements $(e_{k,2i,n})_{k=0}^{p}$ with $n \geq s$, except $e_{k,2j,s}$, where it equals 1.

In the same lexicographical order as above, we arrange all functionals $(\eta_{k,1,0})_{k=0}^{p}$ and $(\eta_{k,2j,s})_{k=0,j=1,s=1}^{p,2^{s-1},\infty}$ into the sequence $(\eta_n)_{n=1}^{\infty}$.

Our next goal is to express the sum $S_N(f) := \sum_{n=1}^{N} \eta_n(f) \cdot f_n$ in terms of the Taylor polynomials of the function $f$. Clearly, $S_N(f) = T_0^{N-1} f$ for $1 \leq N \leq p+1$.

Suppose $p + 2 \leq N \leq 2(p+1)$. Then $S_N(f) = T_0^p f$ on $I_{1,1}$. On the interval $I_{2,1}$, we obtain $S_N(f) = T_0^p f + \sum_{n=p+2}^{N} \eta_{n-p-2,2,1}(f) \cdot e_{n-p-2,2,1}$. For the second term, we have $\sum_{k=0}^{N-p-2} \left[\xi_{k,2,1}(f) - \sum_{m=k}^{p} \xi_{k,2,1}(e_{m,1,0}) \cdot \xi_{m,1,0}(f)\right] \frac{1}{k!}(x - a_{2,1})^k = \sum_{k=0}^{N-p-2} \left[f^{(k)}(a_{2,1}) - \sum_{m=k}^{p} \frac{1}{(m-k)!} a_{2,1}^{m-k} \cdot f^{(m)}(0)\right] \frac{1}{k!}(x - a_{2,1})^k = \sum_{k=0}^{N-p-2} (R_0^p f)^{(k)}(a_{2,1}) \frac{1}{k!} (x - a_{2,1})^k = T_{a_{2,1}}^{N-p-2}(R_0^p f)$.

Therefore, $S_N(f) = T_0^p f$ on $I_{1,1}$ and $S_N(f) = T_0^p f + T_{a_{2,1}}^{N-p-2}(R_0^p f)$ on $I_{2,1}$. Particularly, $S_{2p+2}(f) = T_0^p f + T_{a_{2,1}}^p (R_0^p f) = T_{a_{2,1}}^p f$, by (1). In addition, $S_N^{(k)}(f)(a_{2,1}) = f^{(k)}(a_{2,1})$ for $0 \leq k \leq N - p - 2$, as is easy to check.

Continuing in this way, the values $2p + 3 \leq N \leq 3(p+1)$ correspond to the passage on the interval $I_{2,2}$ from the polynomial $T_0^p f$ to the polynomial $T_{a_{2,2}}^p f$ and the values $3p + 4 \leq N \leq 4(p+1)$ in turn transform $T_{a_{2,1}}^p f$ on $I_{4,2}$ into $T_{a_{4,2}}^p f$.

By the same argument, $S_{2^s(p+1)}(f) = T_{a_{j,s}}^p f$ on $I_{j,s}$ for $1 \leq j \leq 2^s$ and if $j$ with $0 \leq j < 2^s$ is fixed, then the values $N = 2^s(p+1) + j(p+1) + m + 1$ with $0 \leq m \leq p$ transform $T_{a_{j+1,s}}^p f$ on $I_{2j+2,s+1}$ into $T_{a_{2j+2,s+1}}^p f$.

Combining all considerations of this section yields the following result:

**Lemma 1.** *The system $(f_n, \eta_n)_{n=1}^{\infty}$ is biorthogonal. Given $f = (f^{(k)})_{0 \leq k \leq p} \in \prod_{0 \leq k \leq p} C(K(\Lambda))$ and $N = 2^s(p+1) + j(p+1) + m + 1$ with $s \in \mathbb{N}_0$, $0 \leq j < 2^s$, and $0 \leq m \leq p$ we have $S_N(f) = T_{a_{k,s+1}}^p f$ on $I_{k,s+1}$ with $k = 1, 2, \ldots, 2j+1$, $S_N(f) = T_{a_{k,s}}^p f$ on $I_{k,s}$ with $k = j+2, j+3, \ldots, 2^s$, and $S_N(f) = T_{a_{j+1,s}}^p f + T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f)$ on $I_{2j+2,s+1}$.*

## 3. Spaces of differentiable functions and their traces

Let $K$ be a compact subset of $\mathbb{R}$, $p \in \mathbb{N}$. Then the finite product $\prod_{0 \leq k \leq p} C(K)$ equipped with the norm $|(f^{(k)})_{0 \leq k \leq p}|_p = \sup\{|f^{(k)}(x)| : x \in K, k \leq p\}$ is a Banach space. We will consider its subspace $C^p(K)$ consisting of functions on $K$ such that for every nonisolated point $x \in K$ there exist continuous derivatives $f^{(k)}(x)$ of order $k \leq p$ defined in a usual way. If the point $x$ is isolated, then the set $(f^{(k)}(x))_{0 \leq k \leq p}$ can be taken arbitrarily.

The space $\mathcal{E}^p(K)$ of Whitney functions of order $p$ consists of functions from $C^p(K)$ that are extendable to $C^p-$ functions on $\mathbb{R}$. Due to Whitney [18],
$f = (f^{(k)})_{0 \leq k \leq p} \in \mathcal{E}^p(K)$ if

$$(R_y^p f)^{(k)}(x) = o(|x - y|^{p-k}) \quad \text{for } k \leq p \text{ and } x,\ y \in K \text{ as } |x - y| \to 0. \tag{2}$$

The natural topology of a Banach space is given in $\mathcal{E}^p(K)$ by the norm

$$\|f\|_p = |f|_p + \sup\left\{|(R_y^p f)^{(k)}(x)| \cdot |x - y|^{k-p};\ x,\ y \in K, x \neq y, k = 0, 1, \ldots, p\right\}.$$

The Fréchet spaces $C^\infty(K)$ and $\mathcal{E}(K)$ are obtained as the projective limits of the corresponding sequences of spaces. Similarly, the spaces $\mathcal{E}^p(K), \mathcal{E}(K)$ can be defined for $K \subset \mathbb{R}^d$ with $d > 1$.

In general, the spaces $C^p(K)$ and $C^\infty(K)$ contain nonextendable functions and the norms $\|f\|_p$ and $|f|_p$ are not equivalent on $\mathcal{E}^p(K)$. A compact set $K \subset \mathbb{R}^d$ is called Whitney $r$-regular if it is connected by rectifiable arcs, and there exists a constant $C$ such that $\sigma(x, y)^r \le C|x - y|$ for all $x, y \in K$. Here $\sigma$ denotes the intrinsic (or geodesic) distance in $K$. The case $r = 1$ gives the Whitney property $(P)$ [19]. If $K$ is 1-regular, then $C^p(K) = \mathcal{E}^p(K)$ [19, T.1]. A sufficient condition for coincidence $C^\infty(K) = \mathcal{E}(K)$ is $r$-regularity of $K$ for some $r$. For an estimation of $\|\cdot\|_p$ by $|\cdot|_p$ in this case, we refer the reader to [17, IV, 3.11] and [2].

For one-dimensional compact sets we have the following trivial result:

**Proposition 1.** *$C^p(K) = \mathcal{E}^p(K)$ for $2 \le p \le \infty$ if and only if $K = \cup_{n=1}^N [a_n, b_n]$ with $a_n \le b_n$ for $n \le N$.*

**Proof.** Indeed, if $K$ is a finite union of closed intervals, then for any $C^p$-function on $K$ there exists a corresponding extension of the same smoothness, and what is more, the extension which is analytic outside $K$ can be chosen (see e.g. in [12, Cor.2.2.3]).

In the converse case, the complement $\mathbb{R} \setminus K$ contains infinitely many disjoint open intervals. Therefore there exists at least one point $c \in K$ which is an accumulation point of these intervals. Let $K \subset [a, b]$ with $a, b \in K$. Without loss of generality we can assume that $[c, b]$ contains a sequence of intervals from $\mathbb{R} \setminus K$. Then $K \subset K_0 := [a, c] \cup \cup_{n=1}^\infty [a_n, b_n]$ with $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty \subset K, b_1 = b, a_{n+1} \le b_{n+1} < a_n, (b_{n+1}, a_n) \subset \mathbb{R} \setminus K$ for all $n$. Given $1 < p < \infty$, let us take $F = 0$ on $[a, c]$, $F = (a_n - c)^p$ on $[a_n, b_n]$ if $a_n < b_n$. In the case $a_n = b_n$ let $F(a_n) = (a_n - c)^p$ and $F^{(k)}(a_n) = 0$ for all $k > 1$. Thus, $F' \equiv 0$. Then $f = F|_K$ belongs to $C^\infty(K)$, but is not extendable to $C^p$-functions on $\mathbb{R}$ because of violation of (2) for $y = c, x = a_n, k = 0$. $\square$

This nonextendable function can be easily approximated in $|\cdot|_p$ by extendable functions. Therefore, by the open mapping theorem, the following is obtained:

**Corollary 1.** *If $1 < p < \infty$ and $K$ is not a finite union of (maybe degenerated) segments, then the space $(\mathcal{E}^p(K), |\cdot|_p)$ is not complete. The same result is valid for $(\mathcal{E}(K), (|\cdot|_p)_{p=0}^\infty)$.*

It is interesting that the case $p = 1$ is exceptional here.

**Examples.** 1. Let $K = \{0\} \cup (2^{-n})_{n=1}^\infty$. Then $C^1(K) = \mathcal{E}^1(K)$. Indeed, the function $f \in C^1(K)$ is defined here by two sequences $(f_n)_{n=0}^\infty$ and $(f_n')_{n=0}^\infty$ with $\gamma_n := (f_n - f_0) \cdot 2^n - f_0' \to 0$ and $f_n' \to f_0'$ as $n \to \infty$. The second condition gives (2) with $k = 1$. The first condition means (2) with $k = 0, y = 0$. For the remaining case $x = 2^{-n}, y = 2^{-m}$, we have $R_y^1 f(x) = f_n - f_m - f_m'(2^{-n} - 2^{-m}) = \gamma_n \cdot 2^{-n} - \gamma_m \cdot 2^{-m} + (2^{-n} - 2^{-m})(f_0' - f_m')$, which is $o(|2^{-n} - 2^{-m}|)$ as $m, n \to \infty$, since $\max\{2^{-n}, 2^{-m}\} \le 2 \cdot |2^{-n} - 2^{-m}|$. Thus, $f \in \mathcal{E}^1(K)$.

2. Let $K = \{0\} \cup (1/n)_{n=1}^\infty$, $f\left(\frac{1}{2m-1}\right) = 0$, $f\left(\frac{1}{2m}\right) = \frac{1}{m\sqrt{m}}$ for $m \in \mathbb{N}$, and $f' \equiv 0$ on $K$. Then $f \in C^1(K)$, but by the mean value theorem, there is no differentiable extension of $f$ to $\mathbb{R}$.

## 4. Schauder bases in the spaces $C^p(K(\Lambda))$ and $\mathcal{E}^p(K(\Lambda))$

Let us show that the biorthogonal system suggested in Section 2 is a Schauder basis in both spaces $C^p(K(\Lambda))$ and $\mathcal{E}^p(K(\Lambda))$. Here, as before, $p \in \mathbb{N}$. Given $g$ on $K(\Lambda)$, let $\omega(g, \cdot)$ be the modulus of continuity of $g$, that is $\omega(g,t) = sup\{|g(x)-g(y)| : x, y \in K(\Lambda), |x-y| \le t\}, t > 0$. If $x \in I = [a, a+l_s]$, then for any $i \le p$ we have easily

$$|(R_a^p f)^{(i)}(x)| < \omega(f^{(i)}, l_s) + l_s \cdot 2|f|_p \tag{3}$$

and

$$|(R_a^p f)^{(i)}(x)| < 4|f|_p. \tag{4}$$

**Lemma 2.** *The system $(f_n, \eta_n)_{n=1}^{\infty}$ is a Schauder basis in the space $C^p(K(\Lambda))$.*

**Proof.** Given $f \in C^p(K(\Lambda))$ and $\varepsilon > 0$, we want to find $N_\varepsilon$ with $|f - S_N(f)|_p \le \varepsilon$ for $N \ge N_\varepsilon$. Let us take $S$ such that for all $i \le p$ we have

$$3 \cdot \omega(f^{(i)}, l_S) + 14 \cdot l_S \cdot |f|_p < \varepsilon. \tag{5}$$

Set $N_\varepsilon = 2^S(p+1)$. Then any $N \ge N_\varepsilon$ has a representation in the form $N = 2^s(p+1) + j(p+1) + m + 1$ with $s \ge S, 0 \le j < 2^s$, and $0 \le m \le p$. Let us fix $i \le p$ and apply Lemma 1 to $R := (f - S_N(f))^{(i)}(x)$ for $x \in K(\Lambda)$.

If $x \in I_{k,s+1}$ with $k = 1, \ldots, 2j+1$, then $|R| = |(R_{a_{k,s+1}}^p f)^{(i)}(x)| < \varepsilon$, by (3) and (5).

If $x \in I_{k,s}$ with $k = j+2, j+3, \ldots, 2^s$, then $|R| = |(R_{a_{k,s}}^p f)^{(i)}(x)|$ and the same arguments can be used.

Suppose $x \in I_{2j+2,s+1}$. Then $|R| \le |(R_{a_{j+1,s}}^p f)^{(i)}(x)| + |(T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f))^{(i)}(x)|$. For the first term we use (3). The addend vanishes if $m < i$. Otherwise, it is

$$\left| (R_{a_{j+1,s}}^p f)^{(i)}(x) - (R_{a_{j+1,s}}^p f)^{(i)}(a_{2j+2,s+1}) - \sum_{k=i+1}^{m} (R_{a_{j+1,s}}^p f)^{(k)}(a_{2j+2,s+1}) \frac{(x - a_{2j+2,s+1})^{k-i}}{(k-i)!} \right|.$$

Here, we estimate the first and the second terms by means of (3). For the remaining sum, we use (4): $\left|\sum_{k=i+1}^{m} \cdots\right| \le 4|f|_p \sum_{k=i+1}^{m} l_{s+1}^{k-i}/(k-i)! < l_{s+1} \cdot 8|f|_p$. Combining these we conclude that $|R| \le 3(\omega(f^{(i)}, l_s) + l_s \cdot 2|f|_p) + l_{s+1} \cdot 8|f|_p$. This does not exceed $\varepsilon$ due to the choice of $S$. Therefore, $|f - S_N(f)|_p \le \varepsilon$ for $N \ge N_\varepsilon$. □

The main result is given for Cantor-type sets under mild restriction:

$$\exists C_0 : l_s \le C_0 \cdot h_s, \quad \text{for } s \in \mathbb{N}_0. \tag{6}$$

**Theorem 3.** *Let $K(\Lambda)$ satisfy (6). Then the system $(f_n, \eta_n)_{n=1}^{\infty}$ is a Schauder basis in the space $\mathcal{E}^p(K(\Lambda))$.*

**Proof.** Given $f \in \mathcal{E}^p(K(\Lambda))$, we show that the sequence $(S_N(f))$ converges to $f$ as well in the norm $\|\cdot\|_p$. Because of Lemma 2, we only have to check that $|(R_y^p(f - S_N(f)))^{(i)}(x)| \cdot |x-y|^{i-p}$ is uniformly small (with respect to $x, y \in K$ with $x \ne y$ and $i \le p$) for large enough $N$. Fix $\varepsilon > 0$. Due to the condition (2), we can take $S$ such that

$$|(R_y^p f)^{(k)}(x)| < \varepsilon |x-y|^{p-k} \quad \text{for } k \le p \text{ and } x, \ y \in K(\Lambda) \text{ with } |x-y| \le l_S. \tag{7}$$

As above, let $N_\varepsilon = 2^S(p+1)$ and $N = 2^s(p+1) + j(p+1) + m + 1$ with $s \ge S, 0 \le j < 2^s$, and $0 \le m \le p$.

For simplicity, we take the value $i = 0$ since the general case can be analyzed in the same manner. We will consider different positions of $x$ and $y$ on $K(\Lambda)$ in order to show

$$|R_y^p(f - S_N(f))(x)| < C\varepsilon|x - y|^p,$$

where the constant $C$ does not depend on $x$ and $y$. In all cases, we use the representation of $S_N(f)$ given in Lemma 1.

Suppose first that $x, y$ belong to the same interval $I_{k,s+1}$ with some $k = 1, \ldots, 2j + 1$. Then $(f - S_N(f))(x) = R^p_{a_{k,s+1}} f(x)$. From (1) it follows that $R_y^p(f - S_N(f))(x) = R_y^p f(x)$. Here, $|x - y| \le l_{s+1}$, so we have the desired bound by (7).

Similar arguments apply to the case $x, y \in I_{k,s}$ with $k = j + 2, j + 3, \ldots, 2^s$.

If $x, y \in I_{2j+2,s+1}$, then $(f - S_N(f))(x) = (R^p_{a_{j+1,s}} f)(x) - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)(x)$ for $m < p$ and $(f - S_N(f))(x) = (R^p_{a_{2j+2,s+1}} f)(x)$ for $m = p$. Since $R^p(T^m) = 0$ for $m < p$, in both cases we get $R_y^p(f - S_N(f))(x) = R_y^p f(x)$ with $|x - y| \le l_s$ and (7) can be applied once again.

We now turn to the cases when $x$ and $y$ lie on different intervals. Let $x \in I_{k,s+1}, y \in I_{m,s+1}$ with distinct $k, m = 1, \ldots, 2j + 1$. Then $R_y^p(f - S_N(f))(x) = R^p_{a_{k,s+1}} f(x) - \sum_{i=0}^p (R^p_{a_{m,s+1}})^{(i)} f(y)(x - y)^i/i!$. Here, $|x - a_{k,s+1}| \le l_{s+1}$, and $|y - a_{m,s+1}| \le l_{s+1}$; thus, applying (7) gives $|R_y^p(f - S_N(f))(x)| < \varepsilon \cdot l^p_{s+1} + \varepsilon \cdot \sum_{i=0}^p l^{p-i}_{s+1}|x - y|^i/i!$. Now, $|x - y| \ge h_s \ge C_0^{-1} l_s$, by hypothesis. Therefore, $|R_y^p(f - S_N(f))(x)| < C_0^p(e + 1) \cdot \varepsilon \cdot |x - y|^p$, which establishes the desired result. Clearly, the same conclusion can be drawn for $x \in I_{k,s}, y \in I_{m,s}$ with distinct $k, m = j + 2, \ldots, 2^s$, as well for the case when one of the points $x, y$ belongs to $I_{k,s+1}$ with $k \le 2j + 1$ whereas another lies on $I_{m,s}$ with $m = j + 2, \ldots, 2^s$.

It remains to consider the most difficult cases: just one of the points $x, y$ belongs to $I_{2j+2,s+1}$. Suppose $x \in I_{2j+2,s+1}$. We can assume that $y \in I_{2j+1,s+1}$ since other positions of $y$ only enlarge $|x - y|$. Here, $R_y^p(f - S_N(f))(x) = R^p_{a_{j+1,s}} f(x) - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)(x) - \sum_{i=0}^p (R^p_{a_{2j+1,s+1}})^{(i)} f(y)(x - y)^i/i!$. We only need to estimate the intermediate $T^m$ since other terms can be handled in the same way as above. Now, $|T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)(x)| \le \sum_{i=0}^m |(R^p_{a_{j+1,s}})^{(i)} f(a_{2j+2,s+1})|\,|x - a_{2j+2,s+1}|^i/i!$. As before, we use (7). In addition, $|a_{2j+2,s+1} - a_{j+1,s}|$ and $|x - a_{2j+2,s+1}|$ do not exceed $C_0|x - y|$. By that $|T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)(x)| \le C_0^p e\varepsilon|x - y|^p$.

In the last case $x \in I_{2j+1,s+1}, y \in I_{2j+2,s+1}$, we have $R_y^p(f - S_N(f))(x) = R^p_{a_{j+1,s}} f(x) - \sum_{i=0}^p [R^p_{a_{j+1,s}} f - T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)]^{(i)}(y)(x - y)^i/i!$. As above, it is sufficient to consider only $\sum_{i=0}^p [T^m_{a_{2j+2,s+1}}(R^p_{a_{j+1,s}} f)]^{(i)}(y)(x - y)^i/i!$ since for other terms we have the desired bound. Of course, the genuine summation here is until $i = m$. Let us consider a typical term $t_i$ of the last sum. It equals to $(x - y)^i/i! \cdot \sum_{k=i}^m (R^p_{a_{j+1,s}} f)^{(k)}(y)(y - a_{2j+2,s+1})^{k-i}/(k - i)!$. Arguing as above, we obtain $|t_i| \le |x - y|^i/i! \cdot \varepsilon \sum_{k=i}^m l_s^{p-k}\, l^{k-i}_{s+1}/(k - i)! < e\varepsilon|x - y|^i l_s^{p-i}/i!$. By that, $|\sum_{i=0}^m t_i| \le C_0^p e^2\varepsilon|x - y|^p$, which completes the proof. □

**Remarks.** 1. One can enumerate all functions from $(e_{k,1,0})_{k=0}^\infty \cup (e_{k,2j,s})_{k=0,j=1,s=1}^{\infty,2^{s-1},\infty}$ and the corresponding functionals $\eta$ into a biorthogonal sequence $(f_n, \eta_n)_{n=1}^\infty$ in such way that for some increasing sequences $(N_p)_{p=0}^\infty, (q_p)_{p=0}^\infty$ the sum $S_{N_p}(f) = \sum_{n=1}^{N_p} \eta_n(f) \cdot f_n$ coincides with $T^{q_p}_{a_{j,p}} f$ on $I_{j,p}$ for $1 \le j \le 2^p$. Yet, the sequence $(f_n, \eta_n)_{n=1}^\infty$ will not have the basis property in the space $\mathcal{E}(K(\Lambda))$. Indeed, let $F \in C^\infty[0, 1]$ solve the Borel problem for the sequence $(q_n! l_n^{-q_n})_{n=0}^\infty$, that is $F^{(q_n)}(0) = q_n! l_n^{-q_n}$ for $n \in \mathbb{N}_0$ and $F^{(k)}(0) = 0$ for $k \ne q_n$. Let

$f = F|_{K(\Lambda)}$. Then $|f - S_{N_p}(f)|_0 \geq |R_0^{q_p} f(l_p)| \geq \sum_{k=1}^{q_p} f^{(k)}(0) l_p^k / k! - |f(l_p) - f(0)| > 1 - |f(l_p) - f(0)|$. The last expression has a limit 1 as $p \to \infty$, so $S_N(f)$ does not converge to $f$ in $|\cdot|_0$.

For a basis in the space $\mathcal{E}(K(\Lambda))$, see [6].

2. As concerns the paper by Jonsson [9], we note that natural triangulations of the set $K(\Lambda)$ are given by the sequence $\mathcal{F}_s = \{I_{i,s}, 1 \leq i \leq 2^s\}, s \geq 0$. The regularity conditions discussed in [9] are reduced in this case to (6) and

$$\liminf_{s \to \infty} \frac{l_{s+1}}{l_s} > 0. \tag{8}$$

Thus, provided these conditions, the expansion of $f \in \mathcal{E}^p(K(\Lambda))$ with respect to Jonsson's interpolating system converges, at least in $|\cdot|_p$, to $f$, by Proposition 2 in [9]. It is interesting to check the corresponding convergence in topology given by the norm $\|\cdot\|_p$. At the same time it is essential for the proof of by Proposition 2 [9] that the diameters of neighboring triangulations are comparable, which is (8) for Cantor-type sets. Our construction can be applied to any "small" Cantor set with arbitrary fast decrease of the sequence $(l_s)_{s=0}^{\infty}$. The basis problem for the space $\mathcal{E}^p(K(\Lambda))$ in the case of "large" Cantor set with $l_s/h_s \to \infty$ is open.

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